ELECTROMAGNETIC SELF INTERACTION IN STRINGS

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12 March, 1966

Abstract. To facilitate the treatment of electromagnetic effects in applications such as dynamically perturbed vortons, this work employs a covariantly formulated string-source Green measure to obtain a coherent relativistic scheme for describing the self interaction of electromagnetic currents in string models of a very general kind, at leading order in the relevant field gradients, using a regularised gradient operator given by $\widehat{\nabla}_\nu = \overline{\nabla}_\nu + \frac{1}{2}K_\nu$ where $\overline{\nabla}_\nu$ is the usual tangential gradient operator and K_ν is the extrinsic curvature vector.

1 Introduction.

While other kinds of application can be envisaged, the main motivation for most of the work on the development of a relativistic description of electromagnetic self interaction in string models arises from Witten's observation [1] that currents are likely to occur naturally in many conceivable kinds of cosmic strings arising from vortex defects of the vacuum in plausible field theories.

In the first years after Witten's epoch making observation, the importance of electromagnetic effects was actually overestimated in comparison with that of the purely mechanical effects of the current, which were commonly neglected[2]. The fact that the latter would typically be far more important was not generally recognised until after it was shown by Davis and Shellard that "vortons", meaning equilibrium states of string loops, could easily be sustained by the purely mechanical centrifugal effect of a circulating current[3], whereas the magnetostatic support mechanism that had previously been envisaged[4] would usually be too weak. This realisation lead to the development of a more realistic kind of simplification in which electromagnetic effects (other than those arising from a strong but purely external background field) were neglected while attention was concentrated on the suitably non-linear treatment of the purely mechanical effects of the current[5][6].

With adequate means now available for treating the primary mechanical effects of currents in cosmic strings, it is reasonable to reconsider the secondary, but not always negligible, effects due to electromagnetic interaction. A first step towards the inclusion of such effects in a mechanically realistic framework was taken by Peter[7], in a study of circular vorton states. This work confirmed the relative negligibility of the purely magnetostatic support mechanism – due to a spacelike current – that had been considered previously, but on the other hand it also showed that a potentially much more important kind of "spring" effect can be produced by the electrostatic effect of a timelike current.

The purpose of the present work is to provide the machinery needed for generalising Peter's work[7] from strictly stationary states to dynamically perturbed configurations. It will be shown that the kind of "core and sheath" model introduced in early work[4] can be obtained

in a very different manner using the regularisation scheme developed here, which places it on a firmer footing and shows how it can be adapted in such a way as to be fully compatible with the more realistic mechanical models that are now available[8] or that may be introduced in the future to deal for example with string phenomena in which several independent currents may be involved[9].

The present treatment deals only with the leading order in the relevant field gradients. This means that it does not incorporate radiation backreaction, whose inclusion would require extending the scheme to the next differential order, which is left for future work.

2 Overview of the problem.

The primary issue dealt with here is the evaluation of the right hand side of the two dimensional analogue of the familiar relativistic generalisation of Newtons "second law" of motion for a point particle of mass m and charge q, which is given in terms of the unit tangent vector u^μ of the worldline by

$$m a_\mu = q F_{\mu\nu} u^\nu \, , \qquad a^\mu = u^\nu \nabla_\nu u^\mu \, , \tag{1}$$

for an electromagnetic field $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$, where the gravitational field is allowed for via the connection that specifies the Riemannian covariant differentiation operator ∇.

In the case of a string with worldsheet stress energy tensor $\bar{T}^{\mu\nu}$ and electric surface current $\bar{j}^\nu$, the corresponding the general purpose "second law" of extrinsic motion that governs the evolution of its worldsheet is given[5][10] by

$$\bar{T}^{\mu\nu} K_{\mu\nu}{}^\rho = \perp^{\rho\mu} F_{\mu\nu} \bar{j}^\nu \, , \tag{2}$$

where $K_{\mu\nu}{}^\rho$ is the second fundamental tensor of the worldsheet, and $\perp^\rho_{\ \mu}$ is the tensor of orthogonal projection . The latter is defined in terms of the complementary tensor $\eta^\rho_{\ \mu}$ of tangential projection, i.e. the first fundamental tensor, simply by $\eta^\rho_{\ \mu} + \perp^\rho_{\ \mu} = g^\rho_{\ \mu}$, where $g_{\mu\nu}$ is the background spacetime metric. The first fundamental tensor itself is definable as the square $\eta^\rho_{\ \mu} = \mathcal{E}^\rho_{\ \nu} \mathcal{E}^\nu_{\ \mu}$ of the antisymmetric alternating tensor which is specifiable by the formula $\mathcal{E}^{\mu\nu} = 2u^{[\mu} \tilde{u}^{\nu]}$ in terms of (but – except for its orientation – independently of the choice of) an orthonormal worldsheet tangential diad consisting of a timelike unit vector, u^μ say, and the dually related spacelike vector unit vector $\tilde{u}^\mu$, as characterised by $u^\nu u_\nu = -1$, $\tilde{u}^\nu \tilde{u}_\nu = 1$, $u^\nu \tilde{u}_\nu = 0$. The second fundamental tensor is defined in terms of the first one by

$$K_{\mu\nu}{}^\rho = \eta^\sigma_{\ \nu} \overline{\nabla}_\mu \eta^\rho_{\ \sigma} \, , \qquad \overline{\nabla}_\mu = \eta^\nu_{\ \mu} \nabla_\nu \, . \tag{3}$$

As the natural generalisation to two dimensions of the familiar "second law" (1) the extrinsic dynamical equation (2) should in principle be applicable – in the thin limit when higher order longitudinal derivatives are negligible – for any kind of string model, including even the relatively complicated kind that would be needed to describe a warm, dissipatively conducting terrestrial power cable. Like its 1-dimensional analogue (1), the 2-dimensional equation of motion (2) is straightforwardly applicable so long as the field $F_{\mu\nu}$ can be considered to be of purely external origin. However, again like (1), the equation (2) ceases to be so obviously meaningful when one notices that although the locally source free external contribution, $\widetilde{F}_{\mu\nu}$ say, will be well behaved, the self induced contribution $\widehat{F}_{\mu\nu}$ to the total $F_{\mu\nu} = \widehat{F}_{\mu\nu} + \widetilde{F}_{\mu\nu}$ will be divergent in the zero thickness limit.

For a realistic treatment one must recognise that, in a high resolution description, the underlying physical system (which for a cosmic string will be some kind of vortex defect of the

vacuum) will in fact have a finite effective thickness. This will provide a natural "ultraviolet" cut off radius, δ_* say, for a scheme whereby the regularised value of any field is covariantly specified as the average of its values at a distance δ_* in directions orthogonal to the world sheet.

In the point particle case the regularised field tensor $\widehat{F}_{\mu\nu}$ will have the form

$$\widehat{F}_{\mu\nu} = \frac{q}{\delta_*} a_{[\mu} u_{\nu]} \ , \tag{4}$$

in which – as throughout the present discussion – we are neglecting the higher derivative contributions that would be needed to take account of radiation backreaction. If so desired, the self interaction force due to (4) in (1) can be allowed for by transferring the relevant term from the right to the left and then absorbing it into the first term by a mass renormalisation of the usual kind $m \mapsto \widetilde{m}$ where

$$\widetilde{m} = m + \frac{q^2}{2\delta_*} \ . \tag{5}$$

Our purpose here is to carry out the analogous regularisation procedure – and thus to provide the option of a corresponding renormalisation if so desired – for the two dimensional case of a string. In this case the leading "ultraviolet" divergence is not linear but only logarithmic, so that as well as the radius δ_* characterising the effective radius of the string (or to be more specific, that of its current distribution) one also needs a macroscopic "infrared" cut off length, Δ say. It fortunately turns out that no anomally arises in the leading order "ultraviolet" divergence considered here, despite the fact that there is not any way of actually performing the "infrared" cut off in a strictly covariant manner. (It will not be so easy to avoid trouble with Lorentz invariance at the higher order that would be needed for treating radiation backreaction.)

3 Overview of the solution.

The elegantly covariant result one obtains at leading order is as follows. To start with – as was remarked at the outset by Witten[1], and will be made obvious below – the leading contribution to the regularised self field $\widehat{A}_\mu$ on the string world sheet will be expressible (using an appropriate gauge) in terms of of the surface current $\bar{j}^\mu$ there by a simple proportionality relation of the form

$$\widehat{A}_\mu = \widehat{l}\, \bar{j}_\mu \tag{6}$$

in which $\widehat{l}$ is a dimensionless constant of the familiar form

$$\widehat{l} = 2 \ln\{\Delta/\delta_*\} \ . \tag{7}$$

The mathematically non-trivial part of the problem is the derivation of the corresponding regularised field $\widehat{F}_{\mu\nu}$. The worldsheet tangential part can be obtained in the usual way by derivation of $\widehat{A}_\mu$, but since the regularised value is defined only on the worldsheet itself, it cannot be directly differentiated in non tangential directions: $\overline{\nabla}_\nu \widehat{A}_\mu$ is directly meaningful, but $\nabla_\nu \widehat{A}_\mu$ is not. One can however obtain the required result by going through the regularisation procedure again at first differentiated order.

The final result is most conveniently expressible in terms of the self-dual part

$$C_{\mu\nu}{}^{\rho} = \tfrac{1}{2}(K_{\mu\nu}{}^{\rho} + \tilde{\mathcal{K}}_{\mu\nu}{}^{\rho}) \tag{8}$$

of the second fundamental tensor, where the dual is defined by

$$\tilde{\mathcal{K}}_{\mu\nu}{}^{\rho} = \mathcal{E}_{\mu\sigma}\mathcal{E}_{\nu\tau}K^{\sigma\tau\rho} = K_{\mu\nu}{}^{\rho} - \eta_{\mu\nu}K^{\rho} \,, \tag{9}$$

in which K^{ρ} is the trace vector,

$$K^{\rho} = K_{\mu}{}^{\mu\rho} = \overline{\nabla}_{\mu}\eta^{\mu\rho} \tag{10}$$

(whose vanishing, $K^{\mu} = 0$, would express the dynamical equations of motion for a string model of the simplest Goto-Nambu type in the absence of any external force or self interaction). The tensor $C_{\mu\nu}{}^{\rho}$ is identifiable as the conformally invariant *conformation tensor* whose original definition[11], $K_{\mu\nu}{}^{\rho} = C_{\mu\nu}{}^{\rho} + \frac{1}{2}\eta_{\mu\nu}K^{\rho}$, shows that it is trace free, $C_{\mu}{}^{\mu\rho} = 0$.

Although K^{ρ} will be usually be non zero in electromagnetically interacting string models, its contribution cancels out in the final formula for the self field, which reduces, as shown below, simply to

$$\widehat{F}_{\mu\nu} = 2\eta_{[\mu}{}^{\rho}\overline{\nabla}_{\nu]}\widehat{A}_{\rho} + 2\widehat{A}^{\rho}C_{\rho[\mu\nu]} \,. \tag{11}$$

4 Retarded Green measure for a string.

To carry out the first differential order treatment given here (not to mention the second differential order treatment that will need to be carried out in future work to allow for radiation backreaction) the essential tool is the appropriate Green formula for the relevant retarded solution. Since we are concerned here only with the short range "ultraviolet" divergence contribution it will suffice to use the flat space Green formula. (For our present purpose it would make no difference if we used the advanced Green formula: it is only at the next higher differential order that the distinction becomes important.)

For an an isolated particle with charge q the field will be given simply by

$$A^{\mu} = \frac{qu^{\mu}}{r^{\nu}u_{\nu}} \,, \tag{12}$$

where u^{μ} is the future directed unit tangent vector to the trajectory at the (unique) point where it is intersected by the past null cone from the point of interest, and r^{μ} is the past directed null vector – as characterised by $r^{\nu}r_{\nu} = 0$, $r^{\nu}u_{\nu} > 0$ – from this "observation position" to the intersection point on the worldline. What we need is the analogous formula for a string, as provided in the appropriate limit by the well known Liénard-Wiechert formula which solves the source equation in the hyperbolic version

$$\nabla_{\nu}\nabla^{\nu}A^{\mu} = -4\pi\hat{j}^{\mu} \,, \tag{13}$$

(obtained from the gauge invariant version $\nabla_{\nu}F^{\mu\nu} = 4\pi\hat{j}^{\mu}$ by imposition of the Lorentz condition $\nabla_{\nu}A^{\nu} = 0$) for a smooth current distribution $\hat{j}^{\mu}$, in a background with Minkowski metric $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^0)^2$, in the form

$$A^{\mu} = \int\int\int \hat{j}^{\mu}\,\frac{dx^1\,dx^2\,dx^3}{r_0} \,. \tag{14}$$

(Though not manifestly covariant, this formula does in fact give a well defined Lorentz covariant result for a suitably bounded source distribution.)

For the limiting case of a distribution confined to a string, with surface current density $\bar{j}^{\nu}$ that is tangential to the worldsheet, the result will evidently be obtainable as an integral along

the one dimensional curve where the relevant (past or future) null cone from the point under consideration intersects the worldsheet. With respect to a suitably normalised parameter $\mathcal{G}$ this curve will have a tangent vector given by

$$d\bar{x}^\mu = \tilde{r}^\mu \, d\mathcal{G} \, , \qquad \tilde{r}^\mu = \mathcal{E}^{\mu\nu} r_\nu \, , \tag{15}$$

where, as before, $r^\mu = \bar{x}^\mu - x^\mu$ is the *retarded null vector* specifying the ray that goes from the external observation position x^μ to the relevant point $\bar{x}^\mu$ on the worldsheet. It turns out that the dimensionless parameter $\mathcal{G}$ introduced in this way directly specifies the relevant *Green measure*, in terms of which the required analogue of (12) giving the contribution from a finite segment with beginning and end labeled by $-$ and $+$ will be given simply by

$$A^\mu = \int_-^+ \bar{j}^\mu \, d\mathcal{G} \, . \tag{16}$$

The Green measure given by (15) can be seen to be related to the ordinary positive indefinite *proper* length measure ds along the curve by

$$d\mathcal{G} = \frac{ds}{r_\perp} \, , \qquad r_\perp = \sqrt{\tilde{r}^\mu \tilde{r}_\mu} = \sqrt{\perp^{\mu\nu} r_\mu r_\nu} \, . \tag{17}$$

In terms of worldsheet coordinates σ^i and of the corresponding 2-dimensional intrinsic metric given by $\gamma_{ij} = g_{\mu\nu} \bar{x}^\mu{}_{,i} \bar{x}^\nu{}_{,j}$ (using a comma to denote partial differentiation) the fundamental tensor will be given simply by $\eta^{\mu\nu} = \gamma^{ij} \bar{x}^\mu{}_{,i} \bar{x}^\nu{}_{,j}$, so this works out as $d\mathcal{G} = |\gamma^{ij} \bar{x}^\nu{}_{,i} \bar{x}^\rho{}_{,j} r_\nu r_\rho|^{-1/2} \sqrt{\gamma_{hk} \, d\sigma^h \, d\sigma^k}$.

Unlike (15), the explicit formula (17) becomes indeterminate wherever the tangent direction $\tilde{r}^\mu$ is null so that the orthogonally projected distance $r_\perp$ vanishes. This can be overcome at the expense of manifest covariance by referring to a worldsheet frame consisting of any future directed timelike unit tangent vector u^μ, and the dually related spacelike unit vector $\tilde{u}^\mu = \mathcal{E}^{\mu\nu} u_\nu$, which gives the everywhere well behaved expression $d\mathcal{G} = (u_\nu r^\nu)^{-1} \tilde{u}_\mu d\bar{x}^\mu$.

What is commonly quoted in the literature is a more specialised version[12] that is given by $d\mathcal{G} = (\dot{x}^\nu r_\nu)^{-1} \, d\sigma$, but that is valid only for coordinates σ, τ of *conformal* type (meaning that $\dot{x}^\nu \dot{x}_\nu + x'^\nu x'_\nu = 0$, $\dot{x}^\nu x'_\nu = 0$, where $\dot{x}^\nu = \partial \bar{x}^\nu / \partial \tau$, $x'^\nu = \partial \bar{x}^\nu / \partial \sigma$).

5 Regularised gradient operator for self field on worldsheet.

It is almost immediately evident that the leading order contribution to the regularised effective gauge potential on the string will be given by an expression of the form (6). The non-trivial part of the calculation is the evaluation the corresponding effective gradient components as analogously regularised by averaging over an infinitesimal cirle on which $r_\perp = \delta_*$. The terms that remain at leading order give a result of the form

$$\widehat{\nabla_\nu A^\mu} = \widehat{l} \left(\overline{\nabla}_\nu \bar{j}^\mu + \tfrac{1}{2} K_\nu \bar{j}^\mu \right) , \tag{18}$$

using the same regularisation coefficient $\widehat{l}$ as was introduced in (6). For the tangentially differentiated components in the first term, the postulate that this coefficient should be constant ensures that the result of the regularisation procedure will be consistent with what one would get by direct tangential differentiation of the regularised components in the world sheet. This

means that (18) will be expressible by $\widehat{\nabla_\nu A^\mu} = \widehat{\nabla}_\nu \widehat{A}^\mu$ in terms of the *regularised gradient operator* given by

$$\widehat{\nabla}_\nu = \overline{\nabla}_\nu + \tfrac{1}{2} K_\nu \,. \tag{19}$$

With this notation the required self interaction field will be given simply by

$$\widehat{F}_{\mu\nu} = 2\widehat{\nabla}_{[\mu} \widehat{A}_{\nu]} \ . \tag{20}$$

Reorganising this regularised effective field in terms of its distinct purely tangential and mixed components (it has no purely orthogonal one) using the tangentiality property $\perp^\mu{}_\nu \widehat{A}^\nu = 0$ implied by (6), which entails the identity $\perp^\rho{}_\nu \overline{\nabla}_\mu \widehat{A}^\nu = \widehat{A}^\nu K_{\mu\nu}{}^\rho$, one finally obtains the quoted formula (11).

6 Core and sheath representation.

Having obtained the main result (20), or equivalently (11), that was required, one naturally hopes to find that, as in the point particle case, the effect of the ensuing self force contribution will be absorbable into the left hand side of the "second law" equation (2) by the use of an appropriate two dimensional analogue of the renormalisation (5). The concept of such an adjustment was already introduced in the early work of Copeland, Haws, Hindmarsh, and Turok[4] who developped a treatment whereby the current carrying cosmic string of strictly confined "local type" was considered as forming the core of a composite string with an outer sheath of the extended "global type" constituted by its own electromagnetic field. In a compound "core and sheath" string model of this kind, the outer electromagnetic sheath provides an extra contribution to the effective tension and energy per unit length that will be representable by a worldsheet stress energy tensor of the form

$$\widehat{T}^{\mu\nu} = \widehat{l}\ (\bar{j}^\mu \bar{j}^\nu - \tfrac{1}{2} \bar{j}^\rho \bar{j}_\rho \eta^{\mu\nu}) \ , \tag{21}$$

where $\widehat{l}$ is a dimensionless regularisation coefficient of the same form (7) as the one introduced above. Despite their mathematical similarity at the end, it is to be noticed that the physical origin of the coefficient in (6) was very different from that of the one in (21), which came from an integral over a section through the field *outside* the current distribution, whereas the $\widehat{l}$ in (6) comes from an integral with support confined to the current carrying core.

In view of this distinction, it is remarkable that the two approaches turn out in the end to agree perfectly. The form of the self field (11) is easily be seen to be such that the corresponding self force contribution in (2) can indeed be taken over to the left hand side and absorbed into the first term by an adjustment of the form $\bar{T}^{\mu\nu} \mapsto \widetilde{T}^{\mu\nu}$, in which the required two dimensional analogue of the mass renormalisation (5) turns out to have precisely the form that describes the total in the compound core and sheath model, namely

$$\widetilde{T}^{\mu\nu} = \bar{T}^{\mu\nu} + \widehat{T}^{\mu\nu} \,, \tag{22}$$

where the constant coefficient $\widehat{l}$ in (21) is to be identified precisely with the constant coefficient $\widehat{l}$ in (6).

This demonstration that it agrees with the result of the more refined analysis presented here puts the compound "core and sheath" type of model on a firmer footing than hitherto. The agreement is not limited to the prediction of the extrinsic equation of motion discussed above, but still holds just as well for the internal dynamics of the two kinds of model. To see

this, let us consider the dynamical evolution of the loop in an electromagnetic background that may include a locally source free external field contribution $\widetilde{F}_{\mu\nu} = \nabla_{[\mu}\widetilde{A}_{\nu]}$ as well as the self field contribution (11), so that the relevant effective total to be substituted in (2) is

$$F_{\mu\nu} = \widehat{F}_{\mu\nu} + \widetilde{F}_{\mu\nu} \,. \tag{23}$$

The extrinsic dynamical equation (2) is not sufficient by itself to determine the evolution of the string configuration: it is also necessary to know the equations of motion of the surface current $\bar{j}^\mu$ and any other independent internal fields on which the worldsheet stress energy tensor $\bar{T}^{\mu\nu}$ may depend. Whatever the nature of the internal fields, the internal momentum-energy transport equation

$$\eta^\rho{}_\mu \nabla_\nu \bar{T}^{\mu\nu} = \eta^{\rho\mu} F_{\mu\nu}\bar{j}^\nu \tag{24}$$

must be satisfied. Writing the "sheath" contribution (21) in the form

$$\widehat{T}^\nu{}_\mu = \bar{j}^\nu \widehat{A}_\mu - \tfrac{1}{2}\bar{j}^\rho \widehat{A}_\rho \eta^\nu{}_\mu \,, \tag{25}$$

one sees that, subject to the surface current conservation condition

$$\overline{\nabla}_\nu \bar{j}^\nu = 0 \,, \tag{26}$$

the self force contribution on the right in (24) can also be taken over to the left and absorbed into the first term by the same adjustment $\bar{T}^{\mu\nu} \mapsto \widetilde{T}^{\mu\nu}$ as before, where $\widetilde{T}^{\mu\nu}$ is the total for the compound core and sheath model as given by (22). After performing such a transfer both for the extrinsic dynamical equation (2) and the extrinsic dynamical equation (24) the results can be combined in a total force law of the form

$$\overline{\nabla}_\nu \widetilde{T}^\nu{}_\mu = \widetilde{F}_{\mu\nu}\bar{j}^\nu \,, \tag{27}$$

in which only the locally source free purely external field contribution $\widetilde{F}_{\mu\nu}$ is involved on the right hand side.

7 Action formulation for simply conducting case.

What has done so far is valid even for very general string models involving multiple conductivity and finite resistivity as in terrestrial power cables. Let us now restrict attention to the simple strictly conservative kind of model that is appropriate for a Witten type superconducting string, in which the only independent dynamical variable can be taken to be a particle current vector c^μ say, in terms of which the electromagnetic current will be specifiable by a proportionality relation of the form

$$\bar{j}^\mu = q\, c^\mu \,, \tag{28}$$

where q is the relevant mean charge per particle. Such a model will be specified by a *master function*, Λ say, with a a – typically non-linear[8] – dependence on the single scalar variable $\chi = c^\nu c_\nu$, so that its variation determines a corresponding momentum vector given by $\delta\Lambda = \bar{p}_\nu \delta c^\nu$ with $\bar{p}_\mu = \mathcal{K} c_\mu$ where $\mathcal{K} = 2d\Lambda/d\chi$. When self interaction is neglible, the mechanics will be governed just by a Lagrangian density scalar of the simple form

$$\bar{L} = \Lambda + q\, c^\nu A_\nu \,, \tag{29}$$

whose variation determines a gauge dependent total momentum covector: $\delta\bar{L} = \pi_\nu \delta c^\nu$ where

$$\pi_\nu = \bar{p}_\nu + qA_\nu \,. \tag{30}$$

In the application of the variation principle to the corresponding action $\int \bar{L}\sqrt{\|\gamma\|}\, d\sigma\, d\tau$, the current is not an entirely free variable, but must be constrained so as to be conserved, e.g. by taking it to be dual to the gradient of a freely variable stream function ψ say, i.e. $c^\mu = \mathcal{E}^{\mu\nu}\overline{\nabla}_\nu \psi$. For such a model, the worldsheet stress energy tensor reqired for application of the extrinsic equation of motion (2) takes the form

$$\bar{T}^\mu{}_\nu = c^\mu \bar{p}_\nu + (\Lambda - c^\rho \bar{p}_\rho)\eta^\mu{}_\nu \,. \tag{31}$$

It is easy to see that appart from the kinematic particle conservation law $\overline{\nabla}_\nu c^\nu = 0$, whose consequence (26) ensures that the loop will be characterised by a conserved total charge, $Q = \oint \mathcal{E}_{\mu\nu}\,\bar{j}^\nu\, dx^\mu$, the remaining internal dynamical equations for the foregoing model will consist just of the surface integrability condition

$$\eta_{[\mu}{}^\rho \overline{\nabla}_{\nu]}\pi_\rho = 0 \,, \tag{32}$$

which ensures that the the tangentially projected part of the momentum covector (30) is proportional to the surface gradient of a scalar, i.e. $\eta^{\mu\nu}\pi_\nu = \sqrt{\kappa_0}\,\overline{\nabla}^\mu \varphi$, where $\sqrt{\kappa_0}$ is a fixed proportionality factor. It follows that the string loop will be characterised by a conserved circuit integral given by $2\pi N = \oint d\varphi$. The inclusion of the proportionality constant $\sqrt{\kappa_0}$ allows the scalar φ to be adjusted[8] in order to agree with the phase of a boson condensate in an underlying microscopic model, so that it will be periodic with period 2π, in which case N will be quantised as an integral winding number.

It can be seen that when the self interaction is taken into account using the regularisation scheme given above, the foregoing system of equations will still apply provided the field A_μ in the specification (30) of the total momentum per particle is interpreted in accordance with (23) as the combination $A_\mu = \widehat{A}_\mu + \widetilde{A}_\mu$, in which $\widetilde{A}_\mu$ is the well behaved source free external contribution and $\widehat{A}_\mu$ is the regularised self field as given by (6).

The same equations of motion can be obtained by a purely variational approach (something that would not be possible at the next differential order when radiation backreaction will be involved) within the framework of the "core and sheath" model, whose dynamics will be expressible in variational form using an appropriately modified Lagrangian given by $\widetilde{L} = \widetilde{\Lambda} + qc^\nu \widetilde{A}_\nu$ in which only the external field contribution $\widetilde{A}_\mu$ is involved, the self interaction contribution having been absorbed into the uncoupled term by an additive renormalisation $\Lambda \mapsto \widetilde{\Lambda}$ with

$$\widetilde{\Lambda} = \Lambda + \widehat{\Lambda} \,, \qquad \widehat{\Lambda} = \tfrac{1}{2}\,\widehat{l}\; q^2 c^\nu c_\nu \,. \tag{33}$$

The "sheath" contribution, $\widehat{\Lambda} = \frac{1}{2}\widehat{l}\, q^2\chi$, is interpretable as the effective action density of the self generated electromagnetic field, as evaluated[4] by integration across an external section. The effective momentum per particle will undergo a corresponding adjustment $\bar{p}_\mu \mapsto \widetilde{p}_\mu$ with $\widetilde{p}_\mu = \bar{p}_\mu + \widehat{p}_\mu$ where the "sheath" contribution is $\widehat{p}_\mu = \widehat{l}\; q^2 c_\mu$. The corresponding gauge dependent total, $\widetilde{\pi}_\mu = \widetilde{p}_\mu + q\widetilde{A}_\mu$, is the same as was given by (30), i.e. one obtains $\widetilde{\pi}_\mu = \pi_\mu$, so no adjustment is needed for the phase scalar φ or its conserved winding number N.

I wish to thank P. Peter, M. Sakellariadou, and A. Gangui for illuminating discussions.

References

[1] E. Witten, *Nucl. Phys.* **B249**, 557 (1985).

[2] D.N. Spergel, W.H. Press, R.J. Scherrer, *Phys. Rev.* **D39**, 379 (1989).

[3] R.L. Davis, E.P.S. Shellard, *Nucl. Phys* **B323**, 209 (1989).

[4] E. Copeland, D. Haws, M. Hindmarsh, N. Turok, *Nucl. Phys.* **B306**, 908 (1988).

[5] B. Carter, *Phys. Lett.* **B228**, 446 (1989).

[6] B. Carter, *Phys. Lett.* **B238**, 166 (1990).

[7] P. Peter, *Phys. Lett.* **B298**, 60 (1993).

[8] B. Carter, P. Peter, *Phys. Rev.* **D52**, R1744 (1995).

[9] B. Carter, *Nucl. Phys.* **B412**, 345 (1994).

[10] B. Carter, in *Formation and Interactions of Topological Defects* (NATO ASI **B349**), ed. R. Brandenberger, A.-C. Davis, *pp* 304-348 (Plenum, New York, 1995).

[11] B. Carter, *J. Geom. Phys.* **8**, 53 (1992).

[12] E. Copeland, D. Haws, M. Hindmarsh, *Phys. Rev.* **D42**, 726 (1990).